Moment Analysis of Multiplicity Distributions

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Abstract

Moment analysis of global multiplicity distribution previously done for e^+e^- and hh processes is applied to hA and AA collisions. The oscillations of cumulants as functions of their rank are found in all the cases. Some phenomenological approaches and quark-gluon string models are confronted to experimental data. It has been shown that the analysis is a powerful tool for revealing the tiny features of the distributions, and its qualitative results in various processes are rather stable for different multiplicity cut-offs determined by experimental (or Monte-Carlo) statistics.

1 Introduction

Multiplicity distributions of multiparticle reactions contain in the integrated form all the correlations of the system. At the same time they are measured with best accuracy in experiment. Therefore their study is of particular interest. The shapes of the distributions differ drastically for various processes at different energies. Phenomenologically, it is popular to use Poisson (or even sub-Poisson) distributions at lower energies and negative binomial distribution (nowadays, modified negative binomial distribution) at higher energies. The parameters of the distributions vary for different cases and are not determined with high enough precision when fits are done.

However, any multiplicity distribution can be represented not only by the probabilities of n-particle events but also by its moments or by its generating function. It happens that such description is preferred both from the theoretical point of view and for the stability of results obtained from experimental data for various reactions. The solution of QCD equations for

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generating functions predicts [1, 2] the oscillatory behaviour of the cumulant moments of the *parton* distribution in QCD jets as functions of their rank with first minimum located at the point determined by the inverse power of the anomalous dimension γ.

Surprisingly enough, the oscillations of the ratio of cumulant to factorial moments with similar periodicity have been found in experimental *hadron* multiplicity distributions in e^+e^- and $pp(p\bar{p})$ collisions at high energies [3]. No pattern of this kind appears in phenomenological fits (for review see [4]).

In this paper we apply the moment analysis to multiplicity distributions in hadron-nucleus and nucleus-nucleus collisions. The similar (to ee and hh) shape of the above ratio appears in both cases. The only difference is in the depth of the first minimum for different projectiles and targets but not in its location. We compare these results to purely phenomenological fits of the distributions and to some theoretical models (DPM - dual parton model and QGSM - quark-gluon string model). While phenomenological fits do not describe shapes of the ratio obtained from experimental data, the string models reproduce them qualitatively rather well. The minor quantitative difference can be accounted for by slight variation of parameters and should be attributed to experimental uncertainties that demonstrate also how sensitive the method of moment analysis is to tiny features of multiplicity distributions. In that aspect, the qualitative stability of the ratio shapes for reactions with drastically different distributions looks very impressive. The general feature of branching processes inherent both in QCD and in theoretical Monte-Carlo models can be in charge of it.

We have checked that the truncation of multiplicity distributions due to finite statistics and conservation laws does not influence drastically this dependence even though it produces some additional oscillation effect which should be taken into account when comparing models to experimental data. The theoretical arguments in favor of it can be got from studies in paper [5], where it has been shown that conservation laws give rise to corrections of the order of γ^3, while the oscillation effect appears already due to modified leading logarithm approximation terms [1] of the order of γ^2.

Besides, we have found the zeros of the truncated generating functions. Similar to ee and hh processes, they tend to lie close to the circle of unit radius in the complex plane for the high enough truncation limit. Since the highest multiplicities are in AA-collisions, the circle is most clearly seen there. The rightmost zeros tend to the real axis at higher multiplicities that reminds

of Lee-Yang conjecture in statistical physics about the phase transition point to which the rightmost zeros of the grand partition function approach when volume increased. It implies that the singularity of the generating function is very close to the point where all the moments are calculated as derivatives of the generating function. That explains why the moment analysis is so sensitive to slight variations of multiplicity distributions and, at the same time, provides qualitatively similar results for various processes.

2 Moment Analysis of Hadron-Nucleus and Nucleus-Nucleus Collisions

First, let us describe the generalities of the moment analysis. The normalized factorial (F_q) and cumulant (K_q) moments of the multiplicity distribution P_n are defined as

$$F_q = \frac{\langle n(n-1)\dots(n-q+1)\rangle}{\langle n\rangle^q} \equiv \frac{\sum_{n=0}^{\infty} n(n-1)\dots(n-q+1)P_n}{(\sum_{n=0}^{\infty} nP_n)^q}, \qquad (1)$$

$$F_q = \sum_{m=0}^{q-1} C_{q-1}^m K_{q-m} F_m, \qquad (2)$$

where P_n is a probability of n-particle events, q is the rank of the moment, $C_n^m = n!/m!(n-m)!$ are the binomial coefficients.

Thus, if the multiplicity distribution P_n is known one calculates the factorial moments of any rank according to (1), and, afterwards, using the recurrence relations (2) finds out the cumulants.

Theoretically, it is more convenient to start with the generating function

$$G(z) = \sum_{n=0}^{\infty} z^n P_n, \qquad (3)$$

and calculate the moments as its derivatives

$$F_q = \frac{1}{\langle n\rangle^q} \frac{d^q G}{dz^q}\bigg|_{z=1}, \qquad (4)$$

$$K_q = \frac{1}{\langle n\rangle^q} \frac{d^q \ln G}{dz^q}\bigg|_{z=1}. \qquad (5)$$

In practice, the multiplicity distribution is known up to some maximum multiplicity N, and one has to deal with the truncated generating function

$$G_N = \sum_{n=0}^{N} z^n P_n. \tag{6}$$

The interest to the moment analysis of multiplicity data appeared after the solution of QCD equations for the generating function was found [1, 2]. It predicts a special oscillation pattern for the ratio of cumulant to factorial moments

$$H_q = \frac{K_q}{F_q} \tag{7}$$

with first minimum located at $q_{min} \approx 5$ and determined by the inverse value of the QCD anomalous dimension

$$\gamma_0 = (2N_c\alpha_s/\pi)^{1/2}, \tag{8}$$

where α_s is a coupling constant, $N_c = 3$ is the number of colours. For further details see papers [1, 2] and the review paper [4].

Surely, it was the prediction for the moments of *parton* (mostly, gluon) multiplicity distributions. However, when applied to final *hadrons* in e^+e^- and $pp(p\bar{p})$ experimental events [3] the analysis shows the similar structure of the ratio H_q.

We extend this analysis to hadron-nucleus and nucleus-nucleus collisions using experimental data, theoretical Monte-Carlo models and phenomenological fits. We check also how important are various multiplicity cut-offs for the moments. The zeros of the truncated generating function (6) at various cut-offs N are found and discussed.

This analysis is of a special interest because the shapes of multiplicity distributions themselves differ strongly from those in ee and pp collisions.

In Fig.1 we plot the ratio H_q calculated according to the dual parton model [6] and quark-gluon string model [7] for hadron-nucleus and nucleus-nucleus collisions. For a comparison, we show this ratio for charged particles in proton-antiproton collisions at 546 GeV calculated from experimental data and published in [3].

One concludes that the general shapes of H_q are similar for all these processes. The oscillations with first negative minimum in the range $q = 4-6$ are clearly seen. Its depth increases at each step from hh to hA and AA.

The stability of qualitative features of moments is especially remarkable if one compares the multiplicity distributions in all the cases which are very different, indeed.

No phenomenological fit, we are aware of, is able to reproduce such a behaviour. For Poisson distribution all cumulants (and, consequently, ratios H_q) are identically equal to zero (at $q \geq 2$). The modified negative binomial distribution has the generating function

$$G^{(MNBD)}(z) = \left(\frac{1 + \Delta(1-z)}{1 + r(1-z)} \right)^k \tag{9}$$

with three free parameters r, Δ, k related to the average multiplicity by $\langle n \rangle = k(r - \Delta)$. The negative binomial distribution is obtained for $\Delta = 0$. Using (5) it is easy to show that

$$K_q^{(MNBD)} = k^{1-q}(q-1)!(r^q - \Delta^q)/(r - \Delta)^q. \tag{10}$$

Since $k > 0$, the cumulants are always positive for the negative binomial distribution. Recent fits of e^+e^- data [8, 9] give rise to $\Delta < 0$ and $r \leq |\Delta| < 1$. According to (10) it would imply that cumulants change sign at each q being negative at even values of q and positive at odd ones. They do not reveal the broad oscillations shown in Fig.1. The whole picture reminds somewhat the pattern of fixed multiplicity distribution (see [4]). In general, we would like to comment that the fits used in [9] look rather strange since they imply sub-Poissonian distribution of negatively charged particles (with $K_2^{(-)} < 0$), while it is usually claimed that it is super-Poissonian one at high energies. Moreover, with the set of parameters $\Delta = -0.75, r = 0.65$ available in [8, 9] one gets negative probabilities for large n as is easily seen from (9) and was confirmed by computer calculations. Moment analysis of charged particle distributions in e^+e^- data done in [3, 10] has shown the pattern reminding that of $p\bar{p}$ and, according to Fig.1, of reactions with nuclei. The only qualitative difference is the smaller depth of the minimum in ee compared to hadronic reactions.

Contrary to phenomenological fits, the dual parton model [6] and the quark-gluon string model [7] look quite successful in describing the qualitative behaviour of the ratio H_q in hadron-nucleus collisions as shown in Figs.2-4, where the experimental data of NA22 collaboration at 250 GeV have been used [11]. The general periodicity of the curves and the absolute values at

minima and maxima are reproduced rather well even though their positions are somewhat shifted for pAl and KAl. There are several factors which influence the shapes of H_q-curves, and we discuss them in the next section.

3 Asymptotically Subleading Terms of the Moments

The stability of the oscillating pattern in Fig.1 is astonishing because, first, the pattern differs from those of widely used in probability theory and, second, the multiplicity distributions in various processes under consideration are very different and do not seem to have much in common (contrary to H_q-ratios). Moreover, this pattern seems sometimes very sensitive to tiny modifications of multiplicities themselves. This is related to the subtraction procedure used to calculate cumulants according to eq. (2) when factorial moments are known.

However, when comparing to experimental data one should be sure that the proper distributions have been chosen. By that, for example, we mean that the moment analysis of charged multiplicities provides different values of ratios $H_q^{(ch)}$ compared to the values $H_q^{(-)}$ obtained for negatively charged particles. If one considers e^+e^- collisions, the number of charged particles is twice that of negatives. Therefore, the corresponding generating functions are related by

$$G_{ch}(z) = G_-(z^2). \tag{11}$$

Herefrom, one gets for the first five moments:

$$\begin{aligned} F_1^{(ch)} = F_1^{(-)} = K_1^{(ch)} = K_1^{(-)} = 1, \\ F_2^{(ch)} = F_2^{(-)} + \frac{1}{\langle n_{ch}\rangle}, \\ F_3^{(ch)} = F_3^{(-)} + \frac{3}{\langle n_{ch}\rangle} F_2^{(-)}, \\ F_4^{(ch)} = F_4^{(-)} + \frac{6}{\langle n_{ch}\rangle} F_3^{(-)} + \frac{3}{\langle n_{ch}\rangle^2} F_2^{(-)}, \\ F_5^{(ch)} = F_5^{(-)} + \frac{10}{\langle n_{ch}\rangle} F_4^{(-)} + \frac{15}{\langle n_{ch}\rangle^2} F_3^{(-)}, \end{aligned} \tag{12}$$

and the same relations are valid for cumulants (one should just replace F by K in (12)). One concludes that $F_q^{(ch)} > F_q^{(-)}$ at any q, while the inequality $K_2^{(ch)} > K_2^{(-)}$ is always valid for cumulants but $K_3^{(ch)}$ can be less than $K_3^{(-)}$ if the distribution of negatives is sub-Poiisonian one (i.e., if $K_2^{(-)} < 0$). If negative particles are distributed according to Poisson law i.e. $K_q^{(-)} = 0$ for $q \geq 2$, the charged particles dispersion differs from Poissonian. In that case, $K_2^{(ch)} = \langle n \rangle^{-1} > 0$ but higher cumulants are equal to zero according to (12) as if Poisson law is restored just for them.

One can think about this procedure as if the negative particles are created by a "cluster" of negative+positive particle (due to charge conservation they are always produced in pairs). In general, one can consider a cluster model with a definite probability for cluster production and its decay into k particles (see [12], for example). In that case the relation between the generating functions for particles and clusters is

$$G_p(z) = G_c(z^k), \tag{13}$$

and the formulae (12) can be easily generalized to

$$\begin{aligned} F_2^{(p)} &= F_2^{(c)} + \frac{k-1}{\langle n_p \rangle}, \\ F_3^{(p)} &= F_3^{(c)} + \frac{3(k-1)}{\langle n_p \rangle} F_2^{(c)} + \frac{(k-1)(k-2)}{\langle n_p^2 \rangle} \end{aligned} \tag{14}$$

etc. Herefrom, one gets particle distributions wider than cluster ones because $F_2^{(p)} > F_2^{(c)}$ for $k > 1$, e.g., super-Poissonian ones for particles created by Poisson clusters typical for asymptotical multiperipheral and string models. In particular, for the model of Poissonian clusters with $k = 1.4$ charged particles per cluster [12] one gets

$$K_2^{(p)} = \frac{0.4}{\langle n_p \rangle} > 0; \; K_3^{(p)} = -\frac{0.24}{\langle n_p \rangle^2} < 0. \tag{15}$$

To get $K_3^{(p)} > 0$, one needs $k > 2$.

If KNO-scaling holds at asymptotically high energies, all F_q tend to constants and $\langle n_{ch} \rangle \to \infty$. Therefore, all the moments of both distributions coincide at asymptotics. At present energies, however, the correction terms

of the order $O(1/\langle n_{ch}\rangle)$ can be still important, especially, for those cumulants which are close to zero. Therefore, cumulant analysis can give different results for different distributions. Above, we applied it to charged particles multiplicities.

The finite energy of colliding particles and the final experimental statistics truncate the multiplicity distribution at some finite multiplicity N. The relations (12), (14) are independent of this truncation since they are valid for truncated generating functions (6) as well as for total ones (3). However, it influences the values of the moments in these formulae. E.g., the truncated Poisson law for negative particles would give rise to non-zero values of cumulants. Actually, it was conjectured in [13, 14] that at present energies this effect could be rather important for *ee* and *pp* collisions since it imitates the oscillations of cumulants. Surely, it should disappear at higher energies. This is clear both from model calculations and from theoretical arguments of [5] where this effect is of the order of γ^3, while the oscillations in QCD appear already at the modified logarithm approximation of γ^2 order [1, 4]. Nevertheless, at present energies it should be carefully treated for each experiment together with uncertainties imposed by the error bars due to finite statistics. In any Monte-Carlo model they can be reduced by enlarged statistics which is a matter of computer time only.

We do not show the error bars in our Figs but the stability of curves and the regularity of trends shows that they hardly can change our conclusions. For *ee* processes the minima are smaller but even there it was shown [3, 10] that the first minimum is still reliable. We show in Fig.5 that the shapes of H_q-curves in nucleus-nucleus collisions are not very sensitive to the multiplicity cut-off. Truncating the rather flat multiplicity distribution at two different values of N we do not observe any strong shift of the curve and noticeable change of its shape.

4 Zeros of the Truncated Generating Function

The truncated generating function (6) is a polynomial of N-th power with positive coefficients. Therefore, it possesses N complex conjugate zeros in the complex z-plane. We have found their locations in all the above cases

varying the value of N.

The general picture becomes quite stable at large N. The zeros tend to lie close to unit circle in z-plane. The rightmost ones move to the real axis with N increasing as if trying to pinch it at some value of z slightly exceeding 1. They move closer and closer to 1 if we compare ee, hh, hA, AA collisions, correspondingly. Their convergence point implies that the singularity of the total generating function is located near $z = 1$, where all the moments are calculated (see eqs. (4),(5)). It means that moments are very sensitive to the location and the origin of the singularity. (Let us note that the singularity of the negative binomial distribution appears at $z = 1 + r^{-1}$ according to (9), i.e. quite far from $z = 1$ if $r < 1$ as proposed in [8] that contradicts to Figs below).

We demonstrate all these features in Figs.6-9. To show how close are the rightmost zeros to the point $z = 1$ in different processes we give their coordinates x, y and the radii $R = (x^2 + y^2)^{1/2}$ in the Table. Also shown are the maximum multiplicities N. All the radii exceed slightly 1, and the nearest ones to 1 are in nucleus-nucleus collisions.

The above findings could be interesting also if one speculates about the statistical analogies in particle physics [15] in terms of Feynman-Wilson liquid [16, 17]. Then the truncated generating function reminds the grand partition function and the variable z plays the role of fugacity. It has been proven by Lee and Yang that its zeros should lie on the circle in the complex plane and the tendency of rightmost zeros to pinch the positive real axis means that there is a phase transition in the system. Let us note, however, that there is no symmetry in P_n which helped Lee and Yang to prove their statement about zeros location. The similar patterns appearing in all processes in particle physics are quite encouraging for such analogies. The nature of the singularity could be revealed from analysis of experimental data, in principle.

5 Discussion and Conclusions

The moment analysis of multiplicity distributions as applied to hA and AA collisions reveals the oscillation pattern of the ratio H_q similar to that previously found for ee and hh processes. However, the amplitude of oscillations increases for heavier colliding objects, while their periodicity remains stable. Phenomenological fits do not show anything similar to observed patterns.

The string models reproduce them rather well. The multiplicity cut-off is not very important until one comes too close to average multiplicity. The zeros of the truncated generating function tend to be near the unit circle in the complex plane with rightmost zeros moving at high multiplicities toward the real positive axis and pinching it near the point $z = 1$ where derivatives are taken when moments are calculated. This is the singularity point of the total generating function and it is closer to $z = 1$ for heavier colliding objects.

These findings show that there is much more similarity in different collision processes than it could be envisaged from rather different multiplicity distributions. Probably, it is due to the common branching origin of them.

Also, the singularity point of the total generating function would be an interesting topic to learn more about. The density of zeros and their movement to the real axis at higher multiplicities in various processes could indicate its nature.

Since the moments are calculated as derivatives of the generating function at the point very close to the singularity, the moment analysis is powerful and sensitive tool of revealing its structure and tiny features of multiplicity distributions.

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Figure Captions

Fig.1 The ratio H_q for p$\bar{\mathrm{p}}$, hA and AA collisions. (See text for more details).

Fig.2 The ratio H_q for pAl collisions at 250 GeV as calculated from experimental data [11] and according to DPM [6] and QGSM [7].

Fig.3 The ratio H_q for πAl collisions at 250 GeV (experimental data [11] and QGSM [7]).

Fig.4 The ratio H_q for KAl collisions at 250 GeV (experimental data [11] and QGSM [7]).

Fig.5 The ratio H_q for ^{32}S^{238}U collisions at 200A GeV of ^{32}S in the laboratory system, calculated from DPM multiplicity distributions truncated at two values of maximum multiplicity N.

Fig.6 Zeros of the truncated generating function for pAl collisions at 250 GeV.

Fig.7 The same as Fig.6 but for πAl.

Fig.8 The same as Fig.6 but for KAl.

Fig.9 Zeros of the truncated generating function for ^{32}S^{238}U collisions at 200A GeV of ^{32}S in the laboratory system according to DPM [7]

Table

Location of the rightmost zeros of the truncated generating functions for different processes.

collision	R	y	x	N
π Al (QGSM)	1.16042	0.24190	1.1349	41
π Al (exp)	1.16042	0.24771	1.1337	41
K Al (QGSM)	1.12553	0.28954	1.0877	32
K Al (exp)	1.14096	0.30445	1.0996	32
p Al (DPM)	1.15537	0.29461	1.1172	27
p Al (QGSM)	1.09168	0.28592	1.0536	27
p Al (exp)	1.03598	0.31132	0.9881	27
S U (DPM)	1.02265	0.07061	1.0202	80